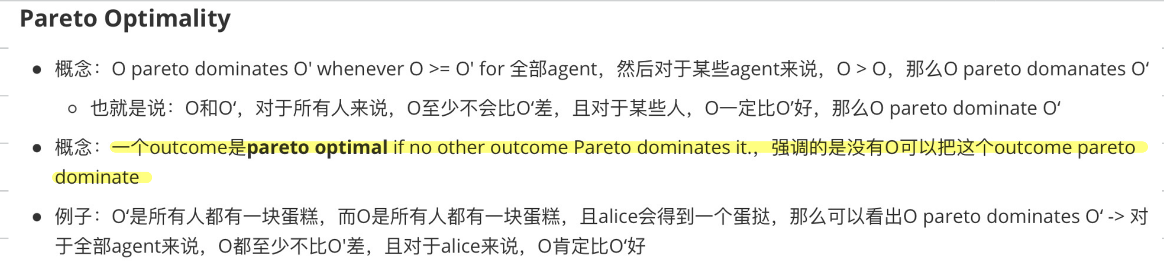
Solution Concept / Strategies

* a formal rule for predicting how a game be played

Pareto Optimally



Pareto Optimality

- 概念：O pareto dominates O' whenever O >= O' for 全部agent，然后对于某些agent来说，O > O，那么O pareto domanates O'
    - 也就是说：O和O'，对于所有人来说，O至少不会比O'差，且对于某些人，O一定比O'好，那么O pareto dominate O'
- 概念：一个outcome是**pareto optimal** if no other outcome Pareto dominates it.，强调的是没有O可以把这个outcome pareto dominate
- 例子：O'是所有人都有一块蛋糕，而O是所有人都有一块蛋糕，且alice会得到一个蛋挞，那么可以看出O pareto dominates O' -> 对于全部agent来说，O都至少不比O'差，且对于alice来说，O肯定比O'好

Can a game have more than one Pareto optimal → Yes

且 every game must have at least one pareto optimal

例：zero sum game 中，任何 outcome 都是 pareto optimal

Nash Equilibrium

Best Responce: $BR_i(a_{-i}) := \{ a_i^* \in A \mid u_i(a_i^*, a_{-i}) \geq u_i(a_i, a_{-i}) \ \forall a_i \in A \}$

↳ 并不是唯一的

Nash Equilibrium · $\forall i \in N: a_i \in BR_i(a_{-i})$

↳ 每个 agent 的行动均是对对方的 best response

A game can have more than one Ps Nash E~

Mixed Strategy Nash Equilibrium:

How to find:

	LW	WL
LW	2,1	0,0
WL	0,0	1,2

设 col play LW 为 p，p2 play WL 为 1-p

令 p1 在 LW和 WL indifferent

$$2p + 0(1-p) = 0p + 1\cdot(1-p)$$

$$2p = 1-p$$

$$p = \frac{1}{3}$$

∴ 当 $P(LW) = \frac{1}{3}$，$P(WL) = \frac{2}{3}$，为 mixed strategy nash

同理可算 row player 的 p

Theorem: Every game with a finite number of players and actions has at least one nash equilibrium

Expected Utility in Mixed Strategy

$$u_i(s) = \sum_{a \in A} \Pr(a|s)\, u_i(a) = \sum_{a \in A} \left(\prod_{j \in N} s_j(a_j) \right) u_i(a)$$

简单来说，先计算出到一个 cell 的 p，再 p(cell) * u_i(cell)，再加起来

例：

		$\frac{2}{3}$	$\frac{1}{3}$
		Ballet	Fight
$\frac{1}{3}$	Ballet	1, 2	0,0
$\frac{2}{3}$	Fight	0,0	2,1

For player1

$$\{\frac{1}{3} \times \frac{2}{3} \times 1\} + \frac{1}{3} \times \frac{1}{3} \times 0 + \frac{2}{3} \times \frac{2}{3} \times 0 + \frac{1}{3} \times \frac{2}{3} \times 2$$

↓

cell [0,0]

Maxmin Strategy

The maxmin strategy for player i is $\arg\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$

maxmin value: $\max_{s_i} \min_{s_{-i}} u(s_i, s_{-i})$

简单来说。就是 当所有别的 player 都对我使出了令我 utility 最低的行动时，我选择一个 max 我的 min 的行动

Min max Strategy

The minmax strategy for player i against player -i is $\arg\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$

player -i 's minmax value is $\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$

简单来说，我默认别的 agent 会做出最能优化他们的决定，提我作出，能最大程度减少它们的最优 utility 的决定

Min max 和 maxmin makes lots of sense in zero sum game.

Theorem: In any finite, two player, zero-sum game, in any Nash equilibrium, each player receives a payoff that is equal to both his max min value and his minmax value

↳ called "value of the game"

Dominant Strategy

For player i, let s_i, s_i' be two strategies of player i, S_{-i} be the set of all strategy profile for the remainning players.

Strictly dominated: $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) \quad \forall s_{-i}$

weakly dominated: $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \quad \forall s_{-i}$

and for at least one s_{-i}: $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$

very weakly dominated $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \quad \forall s_{-i}$

∴ Dominant Strategy: a strategy that dominate any other strategies

More On Max min

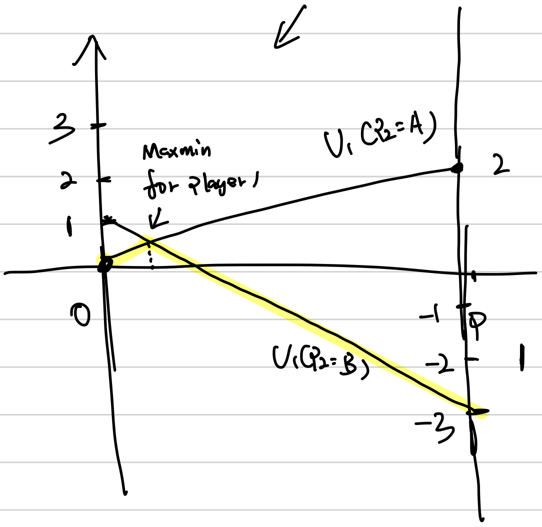
$$\max_{V_1 \in P_1} \min_{S_2 \in S_2} U_1(\bar{u}_1, S_2)$$

Example game:

		A	B
p	A	2, -2	-3, 3
(1-p)	B	0, 0	1, -1

$$U_1(\text{player } 2 = A) = 2p + 0 \cdot (1-p) = 2p$$

$$U_1(\text{player } 2 = B) = -3p + 1(1-p) = -4p+1$$



∴ 对于player 1，当别人作出对自己最坏的打算时，会永远选择下面的线

∴ 若 player1想要最大化降低自己的损失，应该选择整体函数的最高点，即两个U的交点、

Dominant Strategy

可以很轻易地看出，当一个 strategy 被 dominated 时，它不可能出现在 equilibrium 中

∴ 慢慢地把 dominated strategy 给删除依然可以保留 nash ~

iterated removal of dominated strategy

	L	~~C~~	~~R~~
U	3,3	~~0,2~~	~~0,0~~
M	1,3	~~1,1~~	~~5,0~~
D	0,3	~~4,1~~	~~0,0~~

Rationalizability

* 一个 player 是 rationalizable 的 if

这个 player play best response to another player

↓

而 another player 的行动的预测基于 player 的 belief

Correlated Equilibrium

Col

例子 game:

		$\frac{1}{3}$ LW	$\frac{2}{3}$ WL
row	$\frac{2}{3}$ LW	2,1	0,0
	$\frac{1}{3}$ WL	0,0	1,2

$$\therefore EU(row) : \frac{2}{3} \cdot \frac{1}{3} \cdot 2 + \frac{1}{3} \cdot \frac{2}{3} \cdot 1 = \frac{6}{9} : \frac{2}{3}$$

新设定：有一个 correlating device，会分别告诉 row 和 column player 该做什么行为

且这个 device 服从某种分布：$P(WL) = P(LW) = 0.5$

↓ 新的 EU(row) = 1.5 ↓ 能提高大家的幸福

Correleated Equilibrium : device: (V, π, σ)

V : random varicbles , D is respctive domain (respect to each agent)

π : distribution over V

σ : $D_i \longrightarrow A_i$ (告诉 i 要去做什么)

$$\text{Equilibrium}: \sum_{d \in D} \pi(d)\, u_i(\sigma_1(d_1), \ldots \sigma_n(d_n)) \geq \sum_{d \in D} \pi'(d)\, u_i(\sigma_1'(d_1), \ldots \sigma_n(d_n))$$

$\forall \sigma$

Theorem : For every Nash equilibrium σ^* there exists a corresponding correlated equilibrium σ

* Not every correlated equilibrium is equivalent to nash e~

Combination of CE:

1、可以先从 NE开始（NE就是 CE）

2、再加个 random variable 去选择一个 CE

…

Normal Form Game

Formalization: (N, A, u)

N: set of Agents

A: $A_1, \dots, A_n$, action sets for each player

u: $u_1, \dots, u_n$, $u: A \to \mathbb{R}$, utility function

Agents make decision simutaneously

Example:

		player2	
		A	B
player1	A	-1, -1	-3, 0
	B	0, -3	-2, -2

General-sum, 即一个人的利益不等于另一个人的损失

↳ Cooporation 也属于这一类

		player2	
		A	B
player1	A	-1, 1	3, -3
	B	0, 0	-2, 2

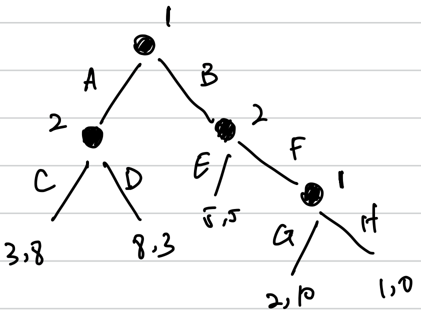
zero-sum. player has opposed interest in anything

Perfect Information Extensive Form Game

Formalization: $G = (N, A, H, Z, \chi, \rho, \delta, u)$

- N : set of players
- A : Single set of Action
- H: set of non torminal choice nodes
- Z: set of terminal choice nodes
- $\chi: H \to 2^A$ action function, 一个 non terminal node 中，有什么 action available
  - ↳ 2^A, 一个 action might be or might not be exise in a non terminal node
- $\rho: H \to N$, the player function, which player's turn
- $\delta: H \times A \to H \cup Z$, successor function，告诉你在一个状态中，执行某个动作的下一个 node
- $u: (u_1, \dots, u_n)$ profile of utility function
  - ↳ $u_i : Z \to \mathbb{R}$ for each player i

例:



Pure Strategy:

Cross product between {all actions} available in each choice node

$\prod_{h \in H} \chi(h)$

例如：对于 player 1:

$\chi(h_1) = \{A, B\}$

$\chi(h_2): \{G, H\}$

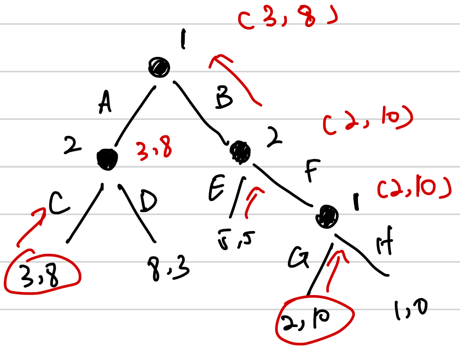
Pure strategies: $\{AG, AH, BG, BH\}$

Pure strategy Equilibrium

↓

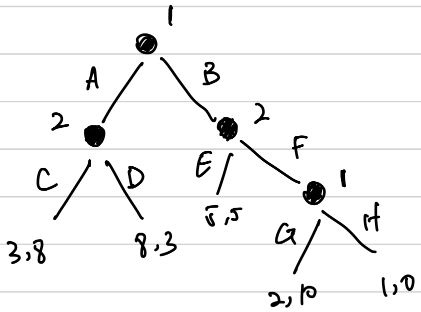
Solve by backword induction；还是要看 normal form

Mixed Strategy: Distribution over Pure Strategies



Induced Normal Form

- * group pure strategies together
- * Each pair of PS identify a terminal node



PS for 1 : {AG, AH, BG, BH}
PS for 2 : {CE, CF, DE, DF}

Normal form :

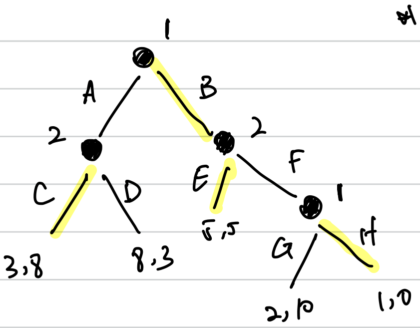
		CE	CF	DE	DF
	AG	3,8	3,8	8,3	8,3
player 1	AH	3,8	3,8	8,3	8,3
	BG	5,5	2,10	5,5	2,10
	BH	5,5	1,0	5,5	1,0

↓

Subgame Nash Equilibria

* Use the same game : Pure strategy Equilibria : BH, CE
AH , CF
AG , CF

对于 BH, CE



* 这个 Equilibria 认为，当 1 get to 第 2 个 choice node 时，它会选 H

but why ?

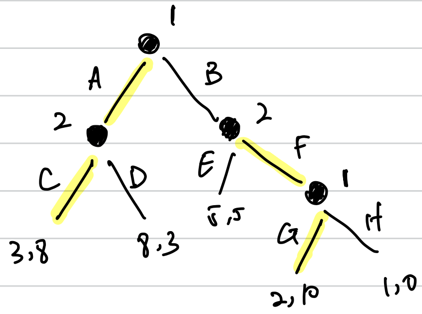
↳ Player 1 threat player 2 that it will choose H

↳ Non credible, 因为 player 1 不会选择 H

Subgame Nash E~ are E~ with no non-credible

Sub game 定义就和 sub-tree 是一样的

Subgame perfect equilibrium is : 当一个 equilibrium 在哪一个 subgame 中都是 equilibrium 时，即为 Sub game perfect



Example: AG, CF

* 任何 backward 推出来的 equilibra 均是 Subgame perfect

Imperfect information Extensive Form Game

Formalization: $G = (N, A, H, Z, \chi, \rho, \sigma, u, I)$

- N : set of players
- A : Single set of Action
- H: set of non terminal choice nodes
- Z: set of terminal choice nodes
- χ: $H \to 2^A$ action function, 一个 non terminal node 中，有什么 action available
  - ↳ 2^A，一个 action might be or might not be exist in a non terminal node
- ρ: $H \to N$，the player function, which player's turn
- σ: $H \times A \to H \cup Z$, successor function，告诉你在一个状态中，执行某个动作的下一个 node
- u: $(u_1, \dots, u_n)$ profile of utility function
  - ↳ $u_i : Z \to \mathbb{R}$ for each player i

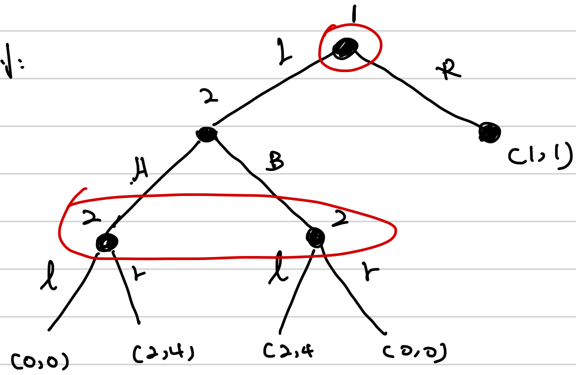
- $I = (I_1, \dots, I_n)$
  - ↳ $I_i = (I_{i,1}, \dots, I_{i,k_i})$ 对于 agent i，它的 I_i 是一个对于 agent i 的 choice node 的一个 partition

    且，对于 agent i，一旦其的 partition 包含一个 partition 中超过一个 h，这些 h 必须：$A(h)$ 一样，$\chi(h) = \chi(h')$

    ☆ 目的是当这些 h 出现一个 partition 中时，player 无法区分他们

    ☆ 每一个 H 都必须在一个 I 中，若 h, h' 是 player 可以区分的，那就把 h 和 h' 放在两个 Information set 中，且 $|I| = 1$

例:



Pure Strategy:

$$\prod_{I_{i,j} \in I_i} \chi(I_{i,j})$$

Cross product for {action available} in each information set

例: $\chi(I_{1,1}) = L, R$

$\chi(I_{1,2}) = l, r$

∴ Pure strategy: Ll, Lr, Rl, Rr

Mixed Strategy:

distribution of pure strategy，要注意的是，在 mixed strategy 的设定中，当选择了一个 PS 后，只能 stick with 这个 PS

Behavioural Strategy

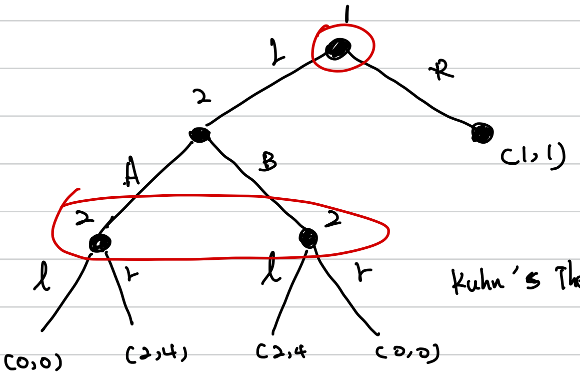
b_i: mapping from an agent's information set

↳ distribution over actions in that info set

例: $I_{1,2}$: action: l, r

$b_i(I_{1,2})$: [0.5: l, 0.5: r]

Behavioural Strategy Vs Mixed Strategy



Example : Ms : [0.6: L l , 0.4: R l]

BS: [0.6 : L, 0.4:R] → $I_{1,1}$

[0.5 : l, 0.5 : r] → $I_{1,2}$

Equivalent between MS and BS

Kuhn's Theorem. Equivalent if induce same distribution over outcome

Important for computation

BS → MS : BS : [0.6 : L, 0.4:R] → $I_{1,1}$

[0.5 : l, 0.5 : r] → $I_{1,2}$

⇓

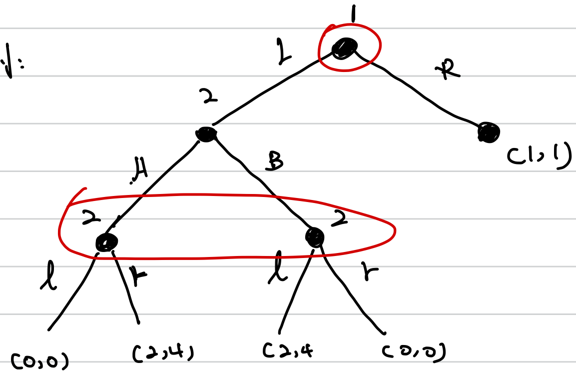
0.3 : L l , 0.3 : L r , 0.2 : R l , 0.2 : R r

MS → BS : ✓ 0.3 : L l , 0.3 : L r , 0.2 : R l , 0.2 : R r

Normal Form of Imperfect information ~

就是把 pure strategy 展开，列成表格 → Any pair of pure strategies identify a Terminal node

例:



→

	A	B
L l	0,0	2,4
L r	2,4	0,0
R l	1,1	1,1
R r	1,1	1,1

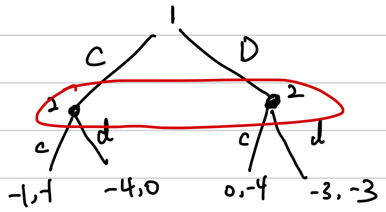
Mapping between Games

- * Perfect information → Imperfect information
  简单，令全部的 H 都分别在一个独立的 information set 中

- * Any normal form game → Imperfect information
  理由很简单，只需把行动者归约到 information set 就好. Such that 他们无法得知自己在哪. i.e. 不知对方做了什么行动

	c	d
C	-1, -1	-4, 0
D	0, -4	-3, -3

=>



Perfect Recall

↳ Never lose information

Player 之所以分不清，不是因为没记住，恰恰是因为，记住了，但 Path 是一模一样的（没有别的信
令它们区分 h）

For any 2 $h \to (h, h')$ in a information set

Path to h: $h_0, a_0, h_1, a_1 \dots h_n, h$

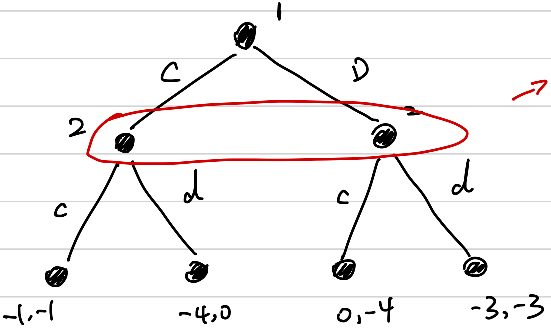
Path to h': $h'_0, a'_0, h'_1, a'_1 \dots h'_m, h'$

① $n = m$

② $h_j, h_j' \in I'$（路径上的 h 都属于 information set such that 也无法被区分）

③ $a_j = a_j'$（路径上的 action 无法被区分）

例子 for Perfect Recall：

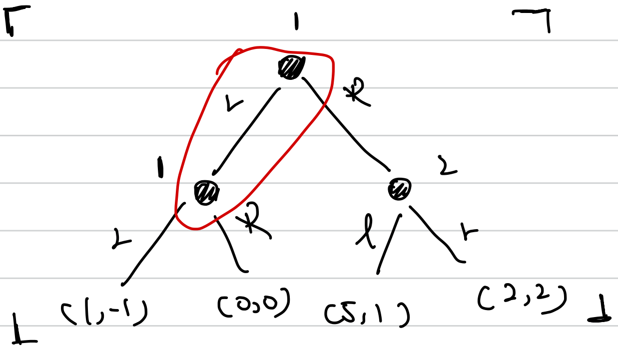


→ In this case, $I_{2,1}$ 就是 root

Imperfect Recall：

player don't remember

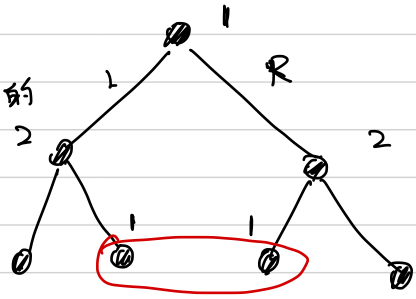
① ↦ in a sense that 路径上不同 sequence 的 H 居然在一个 info set



①

②

虽然经历了不同的
路径，但依然 can't
tell



但在图①中，无法从 BS：[.5:L, .5R] 转换成 MS

∵ 它里的 PS 是 L, R，而 MS 是 Δ(PS)

∴ LR ≠ BS

Kuhn Theorem： In a game of perfect recall, interchangeable in a sense

any MS ⇄ any BS

that the outcome distributions are the same

Repeated Games

→ A game play for more than one times
  ↳ can be any form, mostly NF
  ↳ called stage game

* Finitely Repeated Game

Example:

	C	D
C	-1, -1	-4, 0
D	0, -4	-3, -3

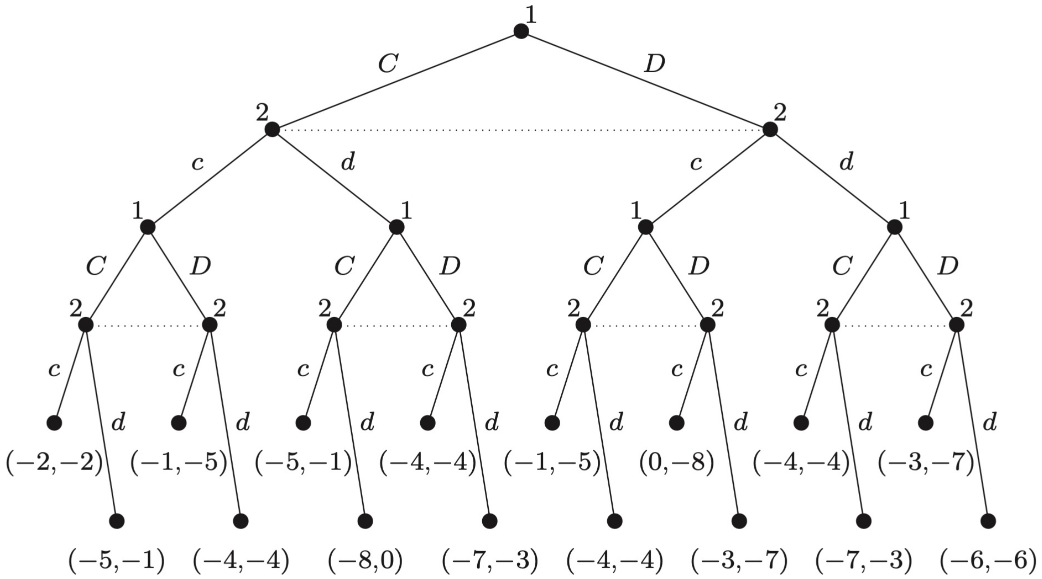
⟹

	C	D
C	-1, -1	-4, 0
D	0, -4	-3, -3

1、在一个 game 中，player 不知道另一个 player 在 player 什么，但只要一个 game 结束了，在另一个 game 之前，player 会知道

2、Pay off 是根据 Σ 每一轮的 payoff

∴ EF might be a better representation



Stationary Strategy: 在每一个 stage game 都用 same strategy

Infinitely Repeated Games

→ 就是 无尽 重复

但 因为是无尽重复，所以一些 本来很 显性的东西，需被重新定义

Pay off:

Average Reward:

Given an infinite sequence of payoff, $r_i^1, r_i^2, \cdots r_i^\infty$

$$\lim_{k\to\infty} \frac{\sum_{j=1}^{k} r_i^{(j)}}{k}$$

Future Discounted reward: Sum of the payoff in the immediate stage game + sum of <u>future rewards</u> (discounted by a constant factor)

Given an infinite sequence of payoff, $r_i^1, r_i^2, \cdots r_i^\infty$

$$r_i^1 + \beta r_i^2 + \beta^2 r_i^3 + \cdots = \sum_{j=1}^{\infty} \underset{\hookrightarrow}{\beta}{}^j r_i^{(j)} \quad \text{discounted factor}$$

Stop: $1-\beta$, intuition是，当 $1-\beta$ 发生的时候. Pay off变 0

Intuition 1: agent cares about current reward more than future reward

Intuition 2: agent also cares about future reward

Pure Strategy: a choice of an action in each staged game → ∞

Fun Strategy: Tit-for-tat : $\text{Coop} \rightarrow \text{defect once} \longrightarrow \text{Coop}$

Trigger : $\text{Coop} \longrightarrow \underset{\substack{\uparrow \\ \text{if opponent defect}}}{\text{defect}} \rightarrow \infty$

Nash equilibria 是 不 possible，但我们可以 找到 在 equibrium中 可以 reach 的 payoff

Folk's Theorem

Intuition : 在一个 infinite game 中 with average reward, 任何在一个 single game 中可以 reach 到的 pair 都可以被 infinite game 中被 reach 到

但这个 pair，这个 r，必须是 enforceable，且 feasible 的

enforceable : $\underbrace{\overset{\text{我们想要的}}{r}: \text{必须} \geq i \text{ 的 } \underline{\text{minmax}}\ (V_i)}$ ↓ 也就是说，当其它全部的 agent 都同时作出对 i 最不利的行为时，i 作出的 best response

↓

intuition 是，player i，不可能作出比 V_i 更差的决定，因为当大家已经作出一个对 i 最差的行为时，what's left 就是 up to player i 了

Feasible : 考虑的是一个 r 是不是可行的，怎么定义？

首先，Average award 是如何定义的：$\lim_{k\to\infty} \frac{\sum_{j=1}^{k} r_i^{(j)}}{k}$ → 也就是说，observe:

1, 0, 1, 0, 1 as sequence of payoffs

∴ Average reward = $\frac{3 \cdot 1 + 2 \cdot \cdot}{5}$

∴ Average award 相当于是每个 stage game 的 payoff 的加权平均

∴ 要让一个 r 是可行的，必须要让 r 可以成为每一个 stage game 的加权平均

∴ 对于一个 desired r，必须 exist αs, such that

$$r_i = \sum_{a\in A} \alpha_a V_i(a)$$

and $\sum_{a \in A} \alpha_a = 1$

例子： Staged game:

		player 2	
		P	NP
player 1	F	-2, 0	3, -2
	T	-2, -1	0, 0

令 r = 0 for player 1

$\alpha_{(0,0)} = 1$, All other α are 1

∴ $\sum_{a \in A} \alpha_a = 1$ and $r_i = \sum_{a \in A} \alpha_a U_i(a)$

只要 r, cab be feasible and enforceable, them r is the payoff profile for some Nash equilibrium of the infinitely repeated G with average rewords

Proof, 令 agents 都 play such that 他们会得到 r，如果不 play 的话，别的 agent 会一起惩罚他

Bayesian Games

Formalization:

① Bayesian Game (N, G, P, I) 根据 I, information set 来确定 Game

N: agents

G: set of Games

P: $\Pi(G) \rightarrow$ probability distribution among games

I: $(I_1, \dots I_N)$, partitions of G

Player 2

		$I_{2,1}$		$I_{2,2}$	
Player 1	$I_{1,1}$	2,0	0,2	2,2	0,3
		0,2	2,0	3,0	1,1
		$p = 0.3$		$p = 0.1$	
	$I_{1,2}$	2,2	0,0	2,1	0,0
		0,0	1,1	0,0	1,2
		$p = 0.2$		$p = 0.4$	

② Bayesian Game (N, A, Θ, P, u)

N: Agents

A: $(A_1, \dots, A_n)$ actions per agent

Θ: $(\Theta_1, \dots \Theta_n)$ types per agents

P: $\Theta \rightarrow [0,1]$ prior over types

u: $(u_1, \dots u_n)$.

↳ u_i: $\underbrace{A \times \Theta}_{} \rightarrow \mathbb{R}$

↳ 一个 agent 的 type + action 决定了他的 utility

同理，要把 definition 2 放入 ① 中 只需要把 I 看作 Θ

Assumption: All possible games have the same $\begin{cases} \text{number of agents} \\ \text{strategy space for each agent} \end{cases}$, only differ in their payoff

Example: θ_{CA_1} : 20% θ_{CA_2} : 80%

θ_{U_1} : 50% θ_{U_2} : 50%

		U_1		U_2	
		S	DS	S	DS
CA_1	T	2, 2	0, 0	1, −1	0, 0
	F	5, 5	0, 0	0, −5	0, 0
		$p = 0.1$		$p = 0.1$	
		S	DS	S	DS
CA_2	T	2, 2	0, 0	1, −1	0, 0
	F	0, 5	0, 0	−5, −5	0, 0
		$p = 0.4$		$p = 0.4$	

Strategy

Pure Strategy : $\theta_i \to A_i$

当一个 agent i 处于 θ_i 的状态时 的必然行动

例: Pure Strategy for CA : T when CA_1 , F when CA_2

F when CA_1 , F when CA_2

F when CA_1 , T when CA_2

T when CA_1 , T when CA_2

	S, DS	DS, DS	DS, S	S, S
TF				
FF				
FT				
TT				

∴ 也可以 induce 一个 normal form

中间填的值是 ex-ante utility

Mixed Strategy : $\theta_i \to \Pi(A_i)$, $s_j(a_j | \theta_j)$

given θ_i for agent, its probality for choosing A_i

例: $s(T | CA_1) = 50\%$, $s(F | CA_1) = 50\%$

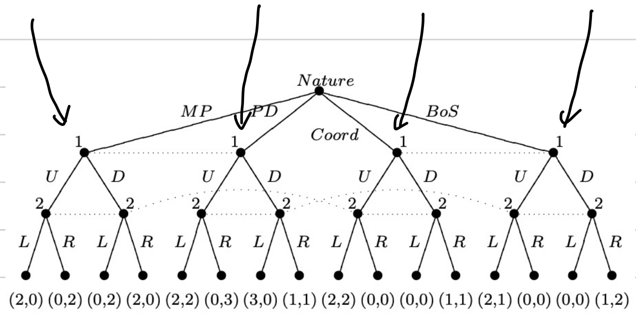
∴ condition on type of the model

Behavioural Strategy: So if we can formalize Bayesian in EF :

	$I_{2,1}$		$I_{2,2}$	
$I_{1,1}$	MP		PD	
	2, 0	0, 2	2, 2	0, 3
	0, 2	2, 0	3, 0	1, 1
	$p = 0.3$		$p = 0.1$	
$I_{1,2}$	Coord		BoS	
	2, 2	0, 0	2, 1	0, 0
	0, 0	1, 1	0, 0	1, 2
	$p = 0.2$		$p = 0.4$	

⇒

Behavioural Strategy is still the distribution over action at each info set



注意看上面的 dash line → information set

Example: θ_{CA_1}: 20% θ_{CA_2}: 80%

θ_{U_1}: 50% θ_{U_2}: 50%

		U_1		U_2	
		S	DS	S	DS
CA_1	T	②, 2	0, 0	1, −1	0, 0
	F	5, 5	0, 0	0, −5	0, 0
		$p = 0.1$		$p = 0.1$	
CA_2	T	2, 2	0, 0	1, −1	0, 0
	F	0, 5	0, 0	−5, −5	0, 0
		$p = 0.4$		$p = 0.4$	

Excepted Utility

* Excepted Utility，之所以是 expected，因为有几个不同的 probability 架接在 pay off 上

Ex-ante (The agent knows nothing about anyone's actual type)

$$EU_i(s) = \sum_{\theta_i \in \Theta_i} p(\theta_i) EU_i(s \mid \theta_i) = \sum_{\theta \in \Theta} p(\theta) \sum_{a \in A} \left(\prod_{j \in N} s_j(a_j \mid \theta_j) \right) u_i(a, \theta)$$

For all θ

简单来说，先用 θ 定位到具体的 game，再算出那个 game 的 expected utility

For example, let pure strategy 为 FT, S,DS，要算 CA 的 Ex-ante

∴ Ex-ante = $P(\theta_{CA_1}, U_1) \cdot U_{CA}(Cf, S), \theta_{CA_1 U_1}) + \cdots$ remaining game

↓

$0.2 \times 0.5 \times 5$

mixed strategy 的话，不用改定位到 θ_i 的 Probability，算那个 game 的 EU 即可

Ex-interim (The agent knows his own type, not other)

$$EU_i(s \mid \theta_i) = \sum_{\theta_{-i} \in \Theta_i} p(\theta_{-i} \mid \theta_i) \sum_{a \in A} \left(\prod_{j \in N} s_j(a_j \mid \theta_j) \right) u_i(a, \theta_{-i}, \theta_i)$$

因为现在已经知道"我"的 type

∴ 在定位 Game 的时候，把 $\theta_{1,2} = P(\theta_1) \times P(\theta_2)$ 改为 $P(\theta_2 \mid \theta_1)$，后面的 EU 不变

例：pure strategy 为 FT, S,DS，已知 CA_1

$P(U_1 \mid CA_1) \cdot U_{CA_1}(Cf, S), \theta_{U_1, CA_1}) + P(U_2 \mid CA_1) \cdot U_{CA_1}(Cf, DS), \theta_{1,2})$

又∵ $P(U_1)$ 和 $P(CA_1)$ 相互独立，$P(U_1 \mid CA_1) = P(U_1)$

∴ $0.5 * 5 + 0.5 * 0 = 2.5$

Ex-post (The agent knows about every one's type)

用这个就更简单，完全知道所有人的 type，只需算各个 game 的 expected utility 即可

$$EU_i(s, \theta) = \sum_{a \in A} \left(\prod_{j \in N} s_j(a_j \mid \theta_j) \right) u_i(a, \theta)$$

Best Response

定义在 ex-ante 上

$$BR_i(s_{-i}) = \underset{s_i' \in S_i}{\arg\max}\ EU(s_i', s_{-i})$$

但这事实上相当于：$\forall_\theta \in \Theta_i$: $BR_i(s_{-i}, \theta) = \underset{s_i, \theta}{\arg\max}\ EU(s_i', s_{-i} \mid \theta)$

↳ perform individual BR calculation for all types of a player

Nash for Bayesian game

$\forall i \quad s_i \in BR_i(s_{-i})$

*也是定义在 Ex-ante 上的

*只要 strategy space for different remain unchanged, BR 都不会变

Ex-post Equilibrium

$\forall \theta, \forall i \quad s_i \in \arg\max_{s_i' \in S_i} EU_i(s_i', s_{-i}, \theta)$

很像 dominant strategy, 但这里 agent 不但不 care 对手的 strategy, 也不 care 任何人的 type

Mechanism

Same, old bayesian game:

Bayesian Game (N, A, Θ, P, u)

N: Agents

A: $(A_1, \dots, A_n)$ actions per agent

Θ: $(\Theta_1, \dots \Theta_n)$ types per agents

P: $\Theta \to [0,1]$ prior over types

u: $(u_1, \dots u_n)$.

$\hookrightarrow u_i: \underbrace{A \times \Theta}_{} \to \mathbb{R}$

↳ 一个 agent 的 type + action 决定了他的 utility

Mechanism:

一个架接在 Bayesian game 上的框 → (A, M)

A: $(A_1, \dots, A_N)$ 规定了各个 agent 的行为

M: $A \to \Pi(O)$ action → distribution of outcomes

目的是, 要如何设计 Mechanism, 从而令 rational agent behave in desired way

- First Price Auction
  - 对于同一个商品，每个bidder写下自己的出价，然后交给拍卖者，拍卖者把货品给出价最高的人。出价最高的人付自己出的价
- Second price auction
  - 对于同一个商品，每个bidder写下自己的出价，然后交给拍卖者，拍卖者把货品给出价最高的人。出价最高的人付第二高的价
- All pay Auction （sealed bid）
  - 对于同一个商品，每个bidder写下自己的出价，然后交给拍卖者，拍卖者把货品给出价最高的人。每个人都要给自己bid的钱

→ Equilibrium: both player bit $\frac{1}{2}$ their true value

↳ 若 n 个 bidder, each should bid $\frac{n-1}{n}$ their true value

→ 不管怎样, 都应该 bid honestly

Revenue Equivalence

只要是一个 auction，有 n 个 risk-neutral agent

且 各个 agent 有一个 Valuation that { private,

independent,

都 draw from $\underset{\downarrow}{\underline{[\underline{v}, \bar{v}]}}$ }

strictly increasing and atomless

那么，任何 mechanism that

1. good 会给出价最高者
2. bid $\underline{v}$ 的人的 EU 是 0

都会得到一样的 expected revenue.

Theorem

Kuhn Theorem: In a game of perfect recall, any mixed strategy of a given agent can be replaced by an equivalent behavioural strategy, and any behavioural strategy, can be replaced by an equivalent mixed strategy.

Zermelo Theorem: Every finite perfect information in EF has at least one pure strategy Nash equilibrium

Every game with a finite number of players and actions has at least one nash equilibrium

In any finite, two player, zero-sum game, in any Nash equilibrium, each player receives a payoff that is equal to both his max min value and his minmax value

Theorem: For every Nash equilibrium σ^* there exists a corresponding correlated equilibrium σ

* Not every correlated equilibrium is equivalent to nash e~

Every (finite) Perfect information EF has a pure strategy nash equilibrium

Level k Example:

Beauty Contest Game

N: 1000 player
Action: Pick a number from 0-100
Payoff: $1000 → player who pick the number closest to the average of all numbers win

Nash

1. People first noticed that any thing above 50 is bad

$\because \frac{100 \times 1000}{1000} / 2 = 50$ (Even if all player bid for 100, the mean is only 50)

2. People then realize that bid at 50 is not optimal

$\because \frac{50 \times 1000}{1000} / 2 = \underline{25}$ (when everyone bid for the maximum possible value 50)

3、 Same for

4. Eventually, people bid for 0

But in reality, people, human being, can't infer that much level

Level k

level 0: bid for uniform distributed value
$\therefore$ mean = 50

level 1 : Think that everyone is L0
$\therefore$ bid for 50

level 2: Think that everyone is L1
$\therefore$ bid for 25

level 3: ~ bid for 12.5

...